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Adaptive notch filter design under multiple identical bandwidths

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ABSTRACT

Digital notch filters are applied to remove or suppress the narrow-band interferences in digital signals, while preserving other components unchanged. Under the condition of the identical notch bandwidths, a novel design method is put forward in this paper to realize the adaptive notch filter with infinite-impulse response (IIR). Firstly, a specially simplified all-pass filter is introduced to construct an IIR non-adaptive notch filter, which serves as the core part in the adaptive one. Secondly, the criterion of least-mean-square (LMS) is applied to design the desired adaptive notch filter (ANF). The designed ANF can track and suppress multiple non-harmonic interference components simultaneously. Finally, the effectiveness and practicability of the proposed design method are verified by a set of experimental results.

1. Introduction

Digital notch filters are used to eliminate or suppress the undesired sinusoidal or narrow-band interferences in digital signals. The applications of notch filters range from power-line noise removal [1–3] into ultra-wide band (UWB) wireless communication systems [4–6], etc. Generally speaking, digital notch filters can be classified into two types: finite-impulse response (FIR) and infinite-impulse response (IIR) notch filters. The former have many better characteristics, including linear phase and system stability, etc. The latter have narrower stop-bands and higher quality factors. In general, an IIR non-adaptive multiple notch filter can eliminate multiple narrow-band components whose center frequencies are known or have been estimated before digital filtering. An IIR adaptive notch filter can track and suppress the sinusoidal interference components whose frequencies are unknown or varying with time.

For non-adaptive multiple notch filters, there are three major design methods: cascade method [7], optimal method [8–10] and all-pass filter based method [11–13]. In the cascade methods, a notch filter is realized by cascading several second-order sections, each of which is a simplest notch filter. However, it may yield unsatisfied distortions in amplitude responses. In the optimal design methods, the poles are placed at the optimal positions to get better amplitude response. The typical method based on quadratic programming was proposed in [9]. In the all-pass filter-based methods, the desired multiple notch filters are constructed by all-pass filters and can be designed by analytical computation techniques. The classical design method was proposed in [11] and widely studied due to the simplicity of solving linear design equations.

Different from non-adaptive notch filters, the adaptive notch filters (ANFs) can track the time-varying center frequencies of narrow-band interference components accurately and suppress them effectively. Consequently, they are widely used for frequency estimation [14–16]. The second-order ANFs [17–19] are very popular due to their simple structures, but they track only one center frequency of narrow-band component in a digital signal. For higher-order ANFs [15,16,20–22], they are able to track and suppress multiple narrow-band interference components simultaneously. The higher-order ANF design method based on least-mean-square (LMS) algorithm [15] was proposed to suppress multiple harmonic components by tracking their fundamental frequency with lower computational complexity.

In this paper, a novel design method of IIR adaptive notch filter with fewer coefficients is proposed to suppress multiple narrow-band components with identical bandwidths. The rest of this paper is organized as follows. Section 2 proposes an improved *non-adaptive* notch filter design method, which is based on a simplified all-pass filter, under the conditions of the identical bandwidths. Section 3 puts forward the novel design method of an *adaptive* notch filter, applied for non-harmonic interference components suppression. Section 4 verifies the effectiveness and practicality

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of the proposed design method by a set of experiments. Finally, conclusions are drawn in Section 5.

2. Simplified design of multiple notch filter

2.1. Problem descriptions

To suppress N sinusoidal or narrow-band interference components in a digital signal, the frequency response of the ideal multiple notch filter is expressed by

$$H_d(e^{j\omega}) = \begin{cases} 0, & \omega = \omega_i \\ 1, & \text{otherwise} \end{cases} \tag{1}$$

where ω_i is the i-th notch frequency, for $i = 1, 2, \ldots, N$. To approximate $H_d(e^{j\omega}), N$ bandwidths $\Delta\omega_1, \Delta\omega_2, \ldots, \Delta\omega_N$ are also used as the specifications of the desired notch filter. In general, the system function of an IIR multiple notch filter is constructed by a rational polynomial. According to the all-pass filter based design method in [11], the polynomial is described by

$$H(z) = \frac{1 + A(z)}{2} \tag{2}$$

where $A(z)$ is a $2N$-order all-pass filter, defined by

$$A(z) = \frac{a_{2N} + a_{2N-1}z^{-1} + \cdots + z^{-2N}}{1 + a_1 z^{-1} + \cdots + a_{2N}z^{-2N}} = \frac{\sum_{k=0}^{2N} a_{2N-k} z^{-k}}{\sum_{k=0}^{2N} a_k z^{-k}} \tag{3}$$

where $a_0, a_1 \ldots, a_{2N} (a_0 = 1)$ are real coefficients. From (2) and (3), the coefficients $a_1, a_2, \ldots, a_{2N}$ must be solved for $A(z)$ and $H(z)$.

Let $z = e^{j\omega} = \cos(\omega) + j\sin(\omega)$, the phase response $\theta_A(\omega) = \angle A(e^{j\omega})$ of $A(z)$ can be calculated by

$$\theta_A(\omega) = -2N\omega + 2\arctan\left[\frac{\sum_{k=1}^{2N} a_k \sin(k\omega)}{1 + \sum_{k=1}^{2N} a_k \cos(k\omega)}\right] \tag{4}$$

Eq. (4) can also be rewritten as a linear equation related with $2N$ coefficients: $a_1, a_2, \ldots, a_{2N}$ (see more details from [11]):

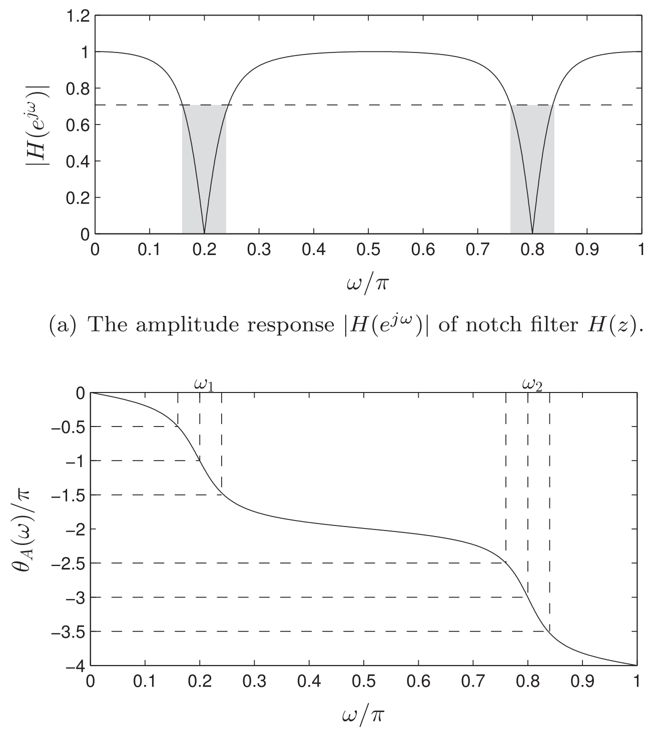
$$\sum_{k=1}^{2N} \{\sin(k\omega) - \tan[\gamma(\omega)]\cos(k\omega)\} a_k = \tan[\gamma(\omega)] \tag{5}$$

where $\gamma(\omega) = [\theta_A(\omega) + 2N\omega]/2$. Due to the special coefficient relationships of the ratio polynomial $A(z)$, the phase response $\theta_A(\omega)$ has the following properties at the i-th notch frequency ω_i with bandwidth $\Delta\omega_i$ (for $i = 1, 2, \ldots, N$)

- At notch frequency $\omega = \omega_i : \theta_A(\omega) = -(2i-1)\pi$.
- At *left* 3-dB frequency $\omega = \omega_i - \Delta\omega_i/2 : \theta_A(\omega) = -(2i-1)\pi + \pi/2$.
- At *right* 3-dB frequency $\omega = \omega_i + \Delta\omega_i/2 : \theta_A(\omega) = -(2i-1)\pi - \pi/2$.

Those relationships between frequencies and phases are interpreted by Fig. 1.

From above discussions, there are $2N$ unknown coefficients in (5) to be solved, hence, at least $2N$ pairs of $(\omega, \theta_A(\omega))$ are required. In [11], $(\omega_i, \theta_A(\omega_i))$ and $(\omega_i - \Delta\omega_i/2, \theta_A(\omega_i - \Delta\omega_i/2))$, for $i = 1, 2, \ldots, N$, are used to construct a $2N$-order linear design equation, under the conditions of N suitable bandwidths. However, if N notch bandwidths are fixed as an identical bandwidth, i.e., $\Delta\omega_i = \Delta\omega_j$ $(1 \leqslant i \neq j \leqslant N)$, the number of the desired coefficients in (5) will be reduced. That is, the design equation will be simplified under the condition of identical bandwidths.



(a) The amplitude response $|H(e^{j\omega})|$ of notch filter $H(z)$.

(b) The phase response $\theta_A(\omega)$ of an all-pass filter $A(z)$.

Fig. 1. The relationship between $|H(e^{j\omega})|$ and $\theta_A(\omega)$ in all-pass filter based notch filter design.

2.2. Simplified design under identical bandwidths

Suppose a multiple notch filter has N notch frequencies $\omega_1, \omega_2, \ldots, \omega_N$ and N equivalent bandwidths $\Delta\omega = \Delta\omega_i$, for $i = 1, 2, \ldots, N$. The relationship between notch bandwidth $\Delta\omega$ and polar radius r can be expressed by [10]:

$$r = \sqrt{\frac{1 - \sin(\Delta\omega)}{\cos(\Delta\omega)}} \tag{6}$$

where $0 < r < 1$, to ensure the stability of filter system.

From (6), the identical bandwidths $\Delta\omega$ means the same polar radius r. According to the coefficients property of $A(z)$ in (3), we can deduct the following relationships (see more details from Appendix A)

$$a_{2N-k} = (r^2)^{N-k} a_k, \quad k = 0, 1, 2, \ldots, N \tag{7}$$

Substituting (7) into (3) yields

$$\begin{aligned} A_s(z) &= \frac{\sum_{k=0}^{N-1} a_{2N-k} z^{-k} + a_N z^{-N} + \sum_{k=N+1}^{2N} a_{2N-k} z^{-k}}{\sum_{k=0}^{N-1} a_k z^{-k} + a_N z^{-N} + \sum_{k=N+1}^{2N} a_k z^{-k}} \\ &= \frac{\sum_{k=0}^{N-1} (r^2)^{N-k} a_k z^{-k} + a_N z^{-N} + \sum_{k=0}^{N-1} a_k z^{k-2N}}{\sum_{k=0}^{N-1} a_k z^{-k} + a_N z^{-N} + \sum_{k=0}^{N-1} (r^2)^{N-k} a_k z^{k-2N}} \end{aligned} \tag{8}$$

where $A_s(z)$ denotes the *special all-pass filter* used to design IIR notch filters with identical bandwidth constraints. Equation (8) can be rearranged as

$$A_s(z) = \frac{\sum_{k=0}^{N-1} a_k [(r^2)^{N-k} z^{-k} + z^{k-2N}] + a_N z^{-N}}{\sum_{k=0}^{N-1} a_k [z^{-k} + (r^2)^{N-k} z^{k-2N}] + a_N z^{-N}} \tag{9}$$

where the coefficient $a_0 = 1$. It is clear that $A_s(z)$ only has N unknown coefficients $a_1, a_2, \ldots, a_N$. Compared with $A(z)$ in (3), the half number of coefficients are only required to design a notch filter with identical bandwidth constraints.

2.3. Non-adaptive notch filter design

The simplified all-pass filter $A_s(z)$ in (9) is used to design a *non-adaptive* multiple notch filter, which will be served as one of the main units of the proposed *adaptive* notch filter in Section 3. Its phase values $\theta_{A_s}(\omega)$ at the notch frequencies $\omega_1, \omega_2, \ldots, \omega_N$, can be calculated as (refer to Section 2.1)

$$\theta_{A_s}(\omega_i) = -(2i-1)\pi, \quad i = 1, 2, \ldots, N \tag{10}$$

Substituting (10) into $\gamma(\omega) = [\theta_{A_s}(\omega) + 2N\omega]/2$ in (5), we obtain

$$\begin{aligned}\tan[\gamma(\omega_i)] &= \tan[N\omega_i - (2i-1)\pi/2] = \tan(N\omega_i - \pi/2)\\ &= -\cos(N\omega_i)/\sin(N\omega_i)\end{aligned} \tag{11}$$

where $i = 1, 2, \ldots, N$. Hence, (5) can be represented as

$$\sum_{k=1}^{2N} a_k \left[\sin(k\omega_i) + \frac{\cos(N\omega_i)}{\sin(N\omega_i)}\cos(k\omega_i)\right] = -\frac{\cos(N\omega_i)}{\sin(N\omega_i)} \tag{12}$$

Multiplying both sides of (12) with $\sin(N\omega_i)$ yields

$$\sum_{k=1}^{2N} a_k \cos[(N-k)\omega_i] = -\cos(N\omega_i) \tag{13}$$

or equivalent equation

$$\begin{aligned}&\left\{\sum_{k=1}^{N-1} a_k \cos[(N-k)\omega_i]\right\} + a_N + \Phi + a_{2N}\cos(N\omega_i)\\ &= -\cos(N\omega_i)\end{aligned} \tag{14}$$

where $\Phi = \sum_{k=N+1}^{2N-1} a_k \cos[(N-k)\omega_i]$ and it can be rewritten as

$$\Phi = \sum_{k=1}^{N-1} a_{2N-k} \cos\{[N-(2N-k)]\omega_i\} = \sum_{k=1}^{N-1} a_{2N-k}\cos[(N-k)\omega_i]$$

From (7), Φ can also be represented as

$$\Phi = \sum_{k=1}^{N-1} r^{2(N-k)} a_k \cos[(N-k)\omega_i] \tag{15}$$

Substituting (15) and $a_{2N} = r^{2N}$ (because $a_0 = 1$) into (14), then

$$\sum_{k=1}^{N-1} a_k (1 + r^{2(N-k)})\cos[(N-k)\omega_i] + a_N = -(1 + r^{2N})\cos(N\omega_i) \tag{16}$$

where $i = 1, 2, \ldots, N$ and $a_1, a_2, \ldots, a_N$ are the coefficients to be solved for $A_s(z)$. Eq. (16) can be written as the matrix-vector form

$$\mathbf{Qa} = \mathbf{b} \tag{17}$$

where the vectors $\mathbf{a}$ and $\mathbf{b}$ are formed by

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}, \quad \mathbf{b} = -(1 + r^{2N}) \cdot \begin{bmatrix} \cos(N\omega_1) \\ \cos(N\omega_2) \\ \vdots \\ \cos(N\omega_N) \end{bmatrix} \tag{18}$$

and according to (16), the entries of matrix $\mathbf{Q} = [q_{ik}]_{N\times N}$ can be defined as

$$q_{ik} = \begin{cases} 1, & k = N \\ (1 + r^{2(N-k)})\cos[(N-k)\omega_i], & \text{otherwise} \end{cases}$$

where $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, N$. Therefore, the desired coefficient vector $\mathbf{a} = [a_1, a_2, \ldots, a_N]^{\mathrm{T}}$ can be solved by $\mathbf{a} = \mathbf{Q}^{-1}\mathbf{b}$. Then, $a_1, a_2, \ldots, a_N$ are used to construct the simplified all-pass filter $A_s(z)$ via (9). After that, the desired notch filter $H(z)$ is obtained by $H(z) = [1 + A_s(z)]/2$. The above design idea of IIR non-adaptive notch filter is summarized in Algorithm 1.

Algorithm 1. The proposed non-adaptive notch filter with identical bandwidths

1: Provide design specifications of a notch filter, including N notch frequencies $\omega_1 < \omega_2 < \cdots < \omega_N$ and identical bandwidth $\Delta\omega$.
2: Calculate polar radius $r (0 < r < 1)$ from the bandwidth $\Delta\omega$ via (6).
3: Construct linear design Eq. (17) through N notch frequencies and the calculated polar radius r.
4: Solve the N-order linear design equation of (17) to obtain the coefficient vector $\mathbf{a} = [a_1, a_2, \ldots, a_N]^{\mathrm{T}}$.
5: Construct all-pass filter $A_s(z)$ via (9) by using the coefficients $a_1, a_2, \ldots, a_N$, and then the desired notch filter via $H(z) = [1 + A_s(z)]/2$.

To verify the effectiveness of Algorithm 1, three notch frequencies are set as $\omega_1 = 0.1\pi, \omega_2 = 0.2\pi, \omega_3 = 0.5\pi$. Their corresponding notch bandwidths are set the same as $\Delta\omega = 0.06\pi$. The amplitude responses of the designed notch filters are shown in Fig. 2, where the *classical* design method is related with $A(z)$ and the *proposed* one is related with $A_s(z)$. In addition, under the different values of $\Delta\omega$, the real gaps between left 3-dB and right 3-dB frequencies and the normalized amplitude errors are listed in Table 1. It is very clear that the proposed method outperforms the classical one discussed in [11].

Moreover, the proposed design method only requires N unknown coefficients, rather than $2N$ ones in the classical one, under identical bandwidth constraints. In addition, when the simplification idea is applied in the adaptive notch filtering, the computational complexity will be reduced significantly.

3. Adaptive notch filter design

From Section 2, when the notch bandwidths are set the same as a constant value, the design equation is simplified in the non-adaptive notch filter design process. That idea can be extended to design an adaptive notch filter, which is sensitive to the computation costs from the filter update process in the adaptive interference components suppression.

3.1. Recursive expression

The all-pass filter $A_s(z)$, shown in (9), can be denoted as $A_s(z) = N_s(z)/D_s(z)$, where $N_s(z)$ and $D_s(z)$ are the numerator and

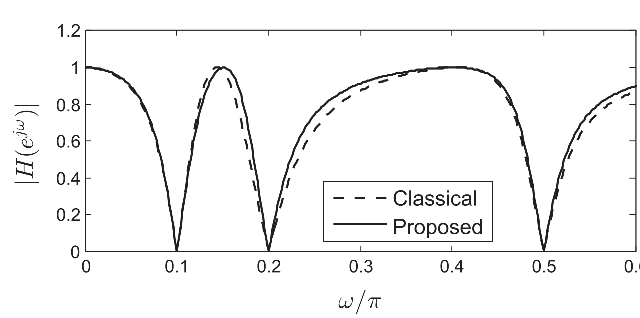


Fig. 2. The amplitude responses of the designed non-adaptive notch filters: $\Delta\omega = 0.06\pi$.

Table 1
Real bandwidths and amplitude response errors under different bandwidth conditions.

$\Delta\omega/\pi$	$\Delta\omega_1/\pi$		$\Delta\omega_2/\pi$		$\Delta\omega_3/\pi$		error[1] $/\pi$	
	Classical	Proposed	Classical	Proposed	Classical	Proposed	Classical	Proposed
0.02	0.0193	0.0201	0.0228	0.0208	0.0217	0.0208	0.0592	0.0574
0.04	0.0362	0.0381	0.0521	0.0436	0.0477	0.0438	0.1181	0.1101
0.06	0.0507	0.0515	0.0877	0.0682	0.0805	0.0701	0.1786	0.1580

[1] The frequency response error is calculated by $\int_0^\pi (1-|H(e^{j\omega})|)^2 d\omega$.

denominator polynomials respectively. Hence, the designed notch filter can be represented as

$$H(z)=\frac{1+N_s(z)/D_s(z)}{2}=\frac{[D_s(z)+N_s(z)]/2}{D_s(z)} \tag{19}$$

where $N_s(z)$ and $D_s(z)$ are related to coefficients $a_1,a_2,\ldots,a_N$. For convenience, we introduce auxiliary coefficients $a_{N+1},a_{N+2},\ldots,a_{2N}$ and $b_0,b_1,\ldots,b_{2N}$ to construct $2N$-order notch filter $H(z)$

$$H(z)=\frac{\sum_{k=0}^{2N}b_kz^{-k}}{1+\sum_{k=1}^{2N}a_kz^{-k}} \tag{20}$$

According to (7), $a_{N+1},a_{N+2},\ldots,a_{2N}$ can be computed by

$$a_{2N-k}=(r^2)^{N-k}a_k,\quad \text{for}\quad k=0,1,\ldots,N-1 \tag{21}$$

where $a_0=1$. According to (9), the coefficients of $D_s(z)$ and $N_s(z)$ meet the reverse relations. Thus, $b_0,b_1,\ldots,b_{2N}$ can be calculated from $a_0,a_1,\ldots,a_N$

$$b_k=b_{2N-k}=\frac{a_k+a_{2N-k}}{2}=\frac{1+(r^2)^{N-k}}{2}\cdot a_k,\quad \text{for}\quad k=0,1,\ldots,N \tag{22}$$

Suppose $x(n)$ and $y(n)$ are the input and output sequences of the notch filter $H(z)$. From (20), we obtain the following differential equation

$$y(n)+\sum_{k=1}^{2N}a_ky(n-k)=\sum_{k=0}^{2N}b_kx(n-k) \tag{23}$$

or equivalently,

$$y(n)=\sum_{k=0}^{2N}b_kx(n-k)-\sum_{k=1}^{2N}a_ky(n-k) \tag{24}$$

where $n>2N$.

From above discussions, $a_1,a_2,\ldots,a_N$ can be solved by (17) or updated in adaptive iterative process. The auxiliary coefficients $a_{N+1},a_{N+2},\ldots,a_{2N}$ and $b_0,b_1,\ldots,b_{2N}$ are calculated from (21) and (22) respectively. Eq. (24) is the recursive expression of the input-output relations of the designed multiple notch filter $H(z)$. In particular, it only involves N coefficients $a_1,a_2,\ldots,a_N$, which must be updated in adaptive notch filtering for unknown narrow-band interference suppression.

3.2. Update process

Suppose the input sequence of a multiple notch filter has N sinusoidal components, and it can be represented as

$$x(n)=\sum_{k=1}^{N}A_k\sin(\omega_kn+\phi_k),\text{for}\quad n=1,2,3,\ldots \tag{25}$$

where A_k,ω_k and ϕ_k are the magnitude, frequency and initial phase of the k-th component, respectively. If all frequencies are estimated and removed accurately, the output sequence of the notch filter will only have zero values. Hence, adaptive notch filter design can be described as an optimization problem

$$\min J(\mathbf{a})\triangleq[y(n)]^2 \tag{26}$$

where $\mathbf{a}=[a_1,\ldots,a_N]^{\mathrm{T}}$ is the real-valued coefficient vector to be solved, $y(n)$ is the output sequence of the notch filter, expressed by (24).

The least-mean-square (LMS) algorithm is relative simple but effective for digital system design. It is also adopted in this paper to obtain the filter coefficient vector $\mathbf{a}$ and then achieve the adaptive notch filtering, whose basic structure is shown in Fig. 3. The update process of LMS algorithm can be represented by

$$\mathbf{a}^{(t+1)}=\mathbf{a}^{(t)}+\mu\cdot\mathbf{s}^{(t)},\quad t=0,1,2,\ldots \tag{27}$$

where μ is the step length, t is the iteration times, $\mathbf{a}^{(t)}$ is the t-th iteration result of $\mathbf{a}$, $\mathbf{s}^{(t)}$ is the t-th search direction (negative gradient direction) and the k-th elements of $\mathbf{s}=[s_1,s_2,\ldots,s_N]^{\mathrm{T}}$ is defined by

$$s_k=-\frac{\partial J(\mathbf{a})}{\partial a_k}=-2y(n)\cdot\beta_k(n),\quad k=1,2,\ldots,N \tag{28}$$

where $\beta_k(n)=\partial y(n)/\partial a_k$ and $n>2N$.

According the relations expressed by (21) and (22), for $a_{N+1},a_{N+2},\ldots,a_{2N}$ and $b_0,b_1,\ldots,b_{2N}$, we obtain

$$\frac{\partial a_{2N-k}}{\partial a_k}=(r^2)^{N-k},\quad \frac{\partial b_{2N-k}}{\partial a_k}=\frac{\partial b_k}{\partial a_k}=\frac{1+(r^2)^{N-k}}{2} \tag{29}$$

where $k=1,2,\ldots,N$. Moreover, according to (24) and (29), $\beta_N(n)=\partial y(n)/\partial a_N$ can be calculated by

$$\begin{aligned}\beta_N(n)=&\frac{\partial b_N}{\partial a_N}\cdot x(n-N)-y(n-N)-\sum_{i=1}^{2N}a_i\frac{\partial y(n-i)}{\partial a_N}\\=&x(n-N)-y(n-N)-\sum_{i=1}^{2N}a_i\beta_N(n-i),\end{aligned} \tag{30}$$

and $\beta_k(n)=\partial y(n)/\partial a_k$ $(k=1,2,\ldots,N-1)$ can be calculated by

$$\begin{aligned}\beta_k(n)=&\frac{\partial b_k}{\partial a_k}\cdot x(n-k)+\frac{\partial b_{2N-k}}{\partial a_k}\cdot x(n+k-2N)\\&-y(n-k)-(r^2)^{N-k}y(n+k-2N)-\sum_{i=1}^{2N}a_i\frac{\partial y(n-i)}{\partial a_k}\\=&\frac{1+(r^2)^{N-k}}{2}[x(n-k)+x(n+k-2N)]-y(n-k)\\&-(r^2)^{N-k}y(n+k-2N)-\sum_{i=1}^{2N}a_i\beta_k(n-i)\end{aligned} \tag{31}$$

Therefore, the search direction $\mathbf{s}^{(t)}$ can be calculated via (24), (30) and (31), where $n>2N$. Particularly, the initial $y(n),\beta_1(n),\beta_2(n),\ldots,\beta_N(n)$ for $n=1,2,\ldots,2N$ are set to be zeros at the beginning of iterations.

The coefficients $\mathbf{a}=[a_1,\ldots,a_N]^{\mathrm{T}}$ are updated by (27) in notch filtering, then the notch filter is constructed by $H(z)=(1+A_s(z))/2$. The block diagram of the proposed adaptive notch filter is shown

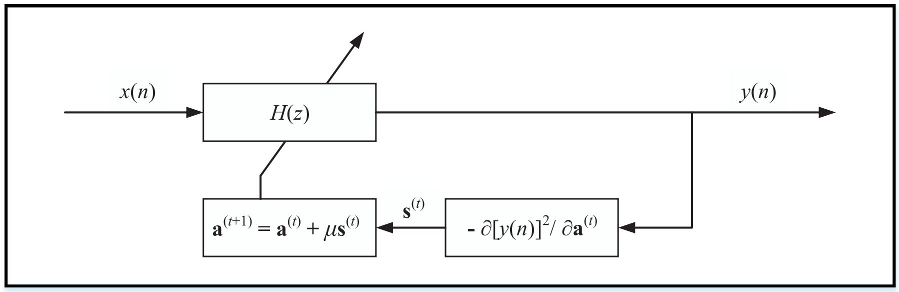


Fig. 3. Structure of the proposed adaptive notch filter.

in Fig. 4. The initial coefficients $\mathbf{a}^{(0)}$ is preset according to the priori information of the input signal (the proposed adaptive notch filter is not sensitive to the initial values). In general, from the priori knowledge of N interference frequencies $\omega_1, \omega_2, \ldots, \omega_N, \mathbf{a}^{(0)} = [a_1, a_2, \ldots, a_N]^{\mathrm{T}}$ can be initialized by Step 1–4 in Algorithm 1.

Algorithm 2. The proposed design method of adaptive notch filter

1: Provide design specifications, including identical bandwidth $\Delta\omega$, iteration step-length μ, notch frequencies number N and input signal $x(n)$ (length: M).
2: Calculate polar radius $r(0 < r < 1)$ through the bandwidth $\Delta\omega$, via (6).
3: Give the initial coefficients $\mathbf{a}^{(0)}$, or estimated through Algorithm 1 with the priori interference frequencies. Afterwards, perform iteratively adaptive notch filtering.
4: **for** $t = 0 : M - 1$ **do**
5: Calculate the auxiliary coefficients via (21) and (22) with the current $\mathbf{a}^{(t)}$.
6: Calculate the output $y(n)$ via (24) and its derivatives $\beta_1(n), \beta_2(n), \ldots, \beta_N(n)$ by (30) and (31).
7: Calculate the search direction $\mathbf{s}^{(t)}$, by substituting $y(n)$ and $\beta_k(n)$ into (28).
8: Update the coefficients vector $\mathbf{a}^{(t+1)}$ via (27).
9: Construct the all-pass filter $A_s(z)$ through (9), and then the current notch filter via $H(z) = [1 + A_s(z)]/2$.
10: **end for**
11: Finally, the output signal $y(n)(n = 1, 2, \ldots, M)$ is obtained.

From above discussions, the proposed design procedure of an adaptive notch filter can be summarized in Algorithm 2. In particular, only N coefficients $a_1, a_2, \ldots, a_N$ are required (rather than $2N$ ones in classical methods) and calculated to construct an adaptive notch filter. It greatly decreases the computational cost in adaptive notch filtering of digital signals. Different from the cascade-based design method using LMS technique in [15], the proposed one is capable to track N different narrow-band components with non-harmonic forms and suppresses them simultaneously.

4. Experiments and analysis

To validate the advantages of the proposed adaptive notch filter design method described in Algorithm 2, we compare it with Steiglitz-McBride method (SMM) based [21] and cascade-based [15] ones. Suppose the input signal $x(n)$ is composed with N harmonic components and a interference component. It can be expressed by

$$x(n) = \sum_{i=1}^{N} \cos(2\pi f_i nT) + v(n) \tag{32}$$

where the sampling frequency $F_s = 1/T = 2000$ Hz, $f_1, f_2, \ldots, f_N$ are the N time-varying frequencies, $v(n)$ is the additive white gaussian noise (AWGN), denoted by $v \sim \mathcal{N}(0, \sigma^2)$. To compare those design methods thoroughly, three cases with changeable frequencies (single, harmonic and non-harmonic) are considered. These frequency values are listed in Table 2.

4.1. Case I: notch filtering of single component

For single component, or $N = 1$ in (32), we assume the frequency f_1 changes from 250 Hz into 300 Hz, then reduces to 280 Hz (refer to Table 2). When $v(n) = 0$ for $n = 1, 2, \ldots, M$, the original signal $x(n)$ and filtered ones (via SMM [21], cascade [15] and proposed methods) are shown in Fig. 5. The tracked frequencies are presented in Fig. 6. Under single component, these three design methods are capable to suppress the sinusoidal component sufficiently with rapid convergence.

4.2. Case II: notch filtering of harmonic components

For harmonic components ($N = 3$), frequencies in (32) meet the harmonic relationships: $f_2 = 2f_1, f_3 = 3f_1$. In our experiments, the value of f_1 changes from 270 Hz to 250 Hz, then returns 270 Hz (details are listed in Table 2). Parameters used in Algorithm 2 include: the polar radius $r = 0.9$, iteration step-length $\mu = 0.05$, notch frequency number $N = 3$, and the input signal length $M = 3000$.

Under noise-free condition, i.e., $v(n) = 0$ for $n = 1, 2, \ldots, M$, the original signal $x(n)$ and filtered signals are shown in Fig. 7. The harmonics are suppressed after short oscillations resulted from fre-

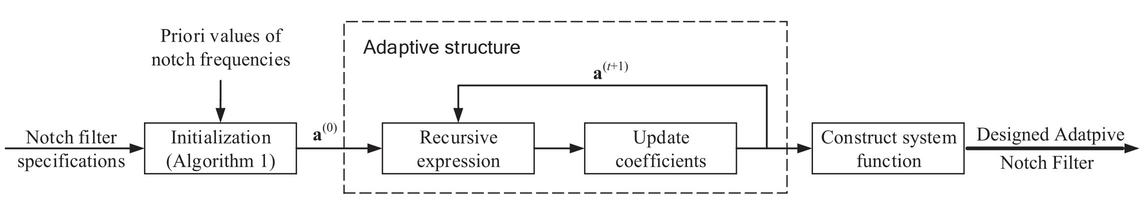


Fig. 4. The block diagram of the proposed adaptive notch filter.

Table 2
Changeable frequencies used in Case I, II and III.

	Case I: single	Case II: harmonic			Case III: non-harmonic		
	f_1	f_1	f_2	f_3	f_1	f_2	f_3
$n = 1 \sim 1000$	250 Hz	270 Hz	540 Hz	810 Hz	200 Hz	400 Hz	700 Hz
$n = 1001 \sim 2000$	300 Hz	250 Hz	500 Hz	750 Hz	150 Hz	450 Hz	750 Hz
$n = 2001 \sim 3000$	280 Hz	270 Hz	540 Hz	810 Hz	200 Hz	400 Hz	800 Hz

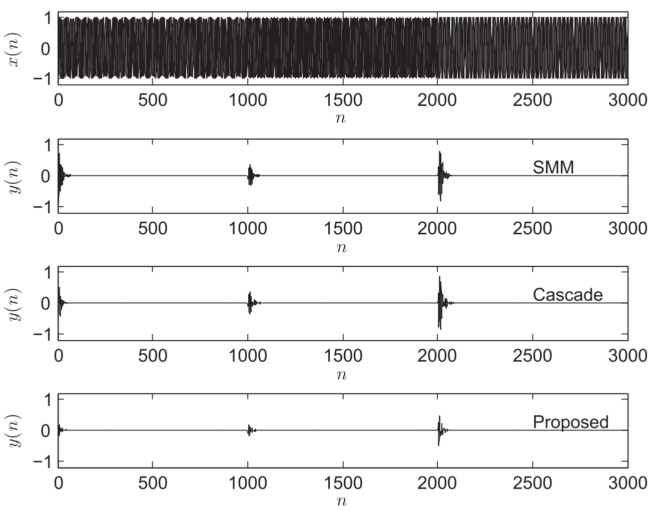


Fig. 5. The original signal $x(n)$ and filtered signals (Case I).

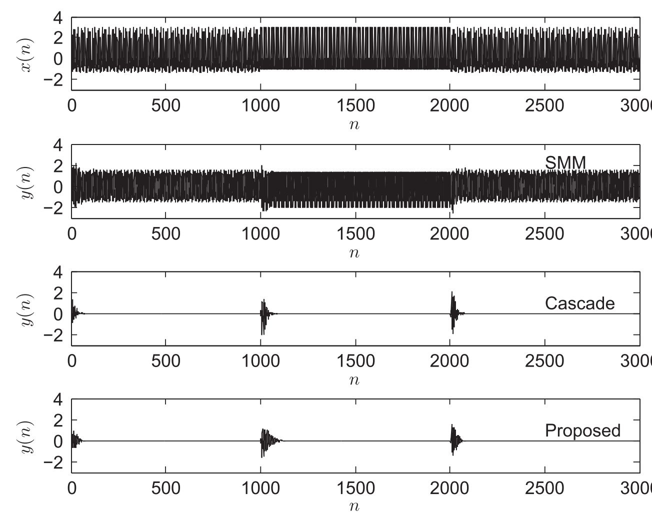


Fig. 7. The original signal $x(n)$ and filtered signals (Case II).

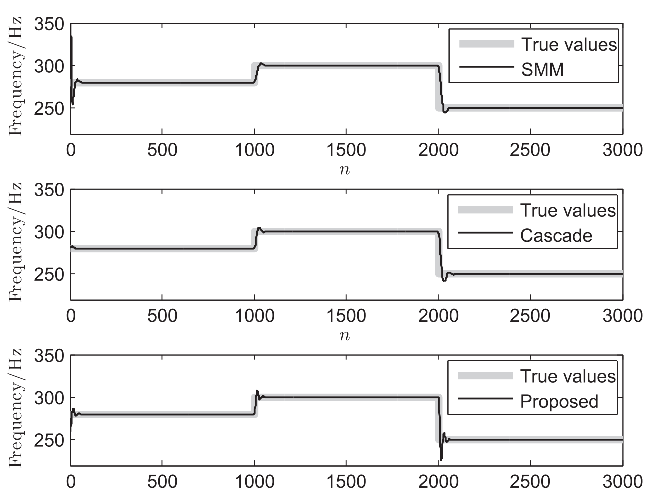


Fig. 6. Tracked frequencies of ANF design methods (Case I).

quency changing suddenly. The tracked frequencies in adaptive notch filtering are shown in Fig. 8. When noise is free, the cascade and proposed methods obtain satisfied performance. However, the cascade method only tracks the fundamental frequency then the other two frequencies are calculated from it, while the proposed one is capable to track the three frequencies simultaneously.

To compare design methods thoroughly, the filtered results are presented under different variance σ (ranging form 0 to 0.7). The root mean square error (RMSE) is introduced to quantify the performance of filter outputs.

$$\mathrm{RMSE} = \sqrt{\frac{\sum_{i=1}^{M} [y(i) - y_d(i)]^2}{M}} \tag{33}$$

where $y(n)$ is the observed output of the notch filter, $y_d(n)$ is the ideal output (all-zeros values), and M is the length of the output, equal to that of the input. When the polar radius r = 0.9 or 0.85, the RMSE values of the cascade and proposed methods are shown in Fig. 9. The proposed method has relatively lower RMSE values, which illustrates the superiority of it.

4.3. Case III: notch filtering of non-harmonic components

In this case, we validate the design performance of the proposed method by using the input signal with non-harmonic frequency components. The input signal has the same construction as (32), but the frequencies f_1, f_2 and f_3 have no harmonic relationships, which are shown in Table 2.

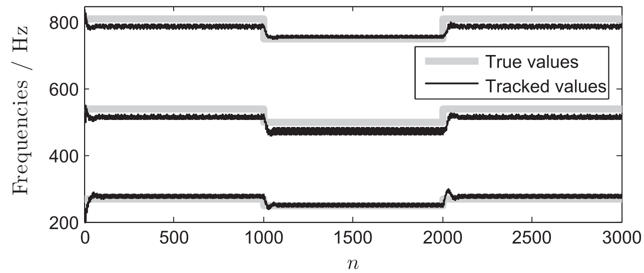
The original signal (when $\sigma = 0$) and adaptive filtered signals are shown in Fig. 10. The tracked frequencies in notch filtering are shown in Fig. 11. Since the cascade method can only handle with the harmonic condition, it is invalid when the interference frequencies are non-harmonic. However, the proposed method is capable to track and suppress multiple non-harmonic frequencies simultaneously. These results validate the effectiveness and competitiveness of the proposed design method.

4.4. Case IV: Noise suppression of ECG signal

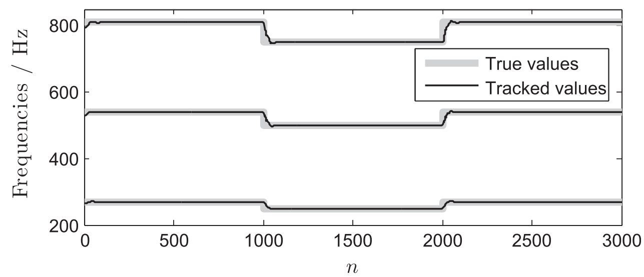
To verify the practicability of the proposed adaptive notch filter, a noisy electrocardiogram (ECG) signal is treated as the input of the notch filter. It is represented by

$$x(t) = x_0(t) + \sum_{i=1}^{3} \cos(2\pi f_i t) \tag{34}$$

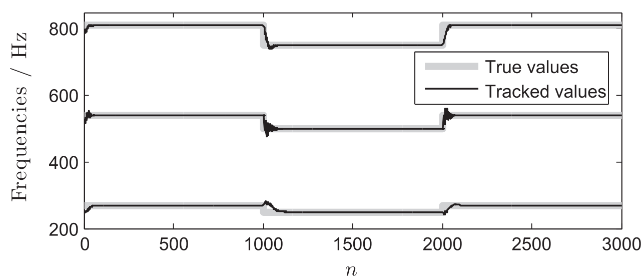
where $x_0(t)$ is the original ECG signal sampled by the frequency $F_s = 360$ Hz. The frequency fluctuations of three interference components are preset and shown in Table 3. Note that those interfer-



(a) The SMM method



(b) The cascade method



(c) The proposed method

Fig. 8. Tracked frequencies of ANF design methods (Case II).

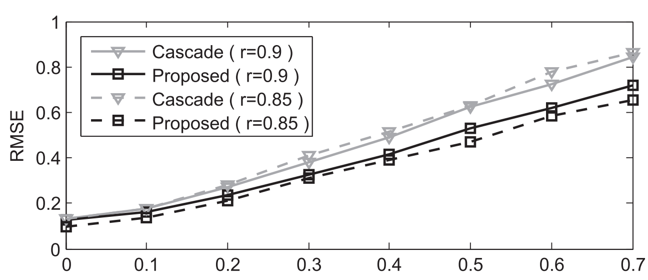


Fig. 9. The RMSE values of the filtered signals (Case II).

ence frequencies are non-harmonic and time-varying (having sudden frequency variations).

The original input ECG signal and filtered one (via the proposed method) are shown in Fig. 12. It is very difficult to obtain any useful information from $x(t)$ directly because of the strong narrowband interferences. After adaptive notch filtering, there are some gradually decaying oscillations follow the frequency hoppings (t = 4 s and 8 s), but the notable interference components have been suppressed significantly. The tracked frequencies in adaptive notch filtering are shown in Fig. 13. It is clear that the estimated notch frequencies are very closed to the predefined values listed in Table 3. It also demonstrates the practicality and effectiveness of the proposed design method.

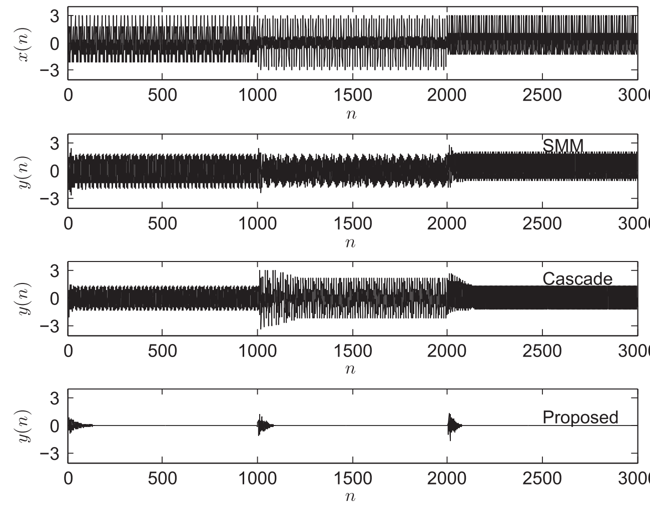
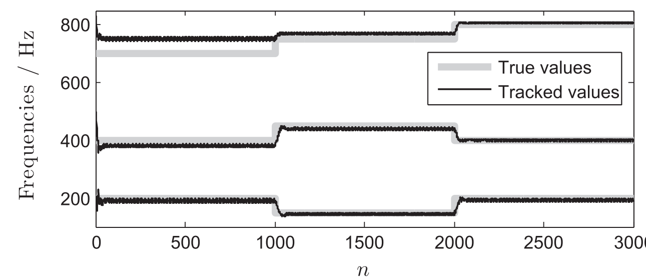
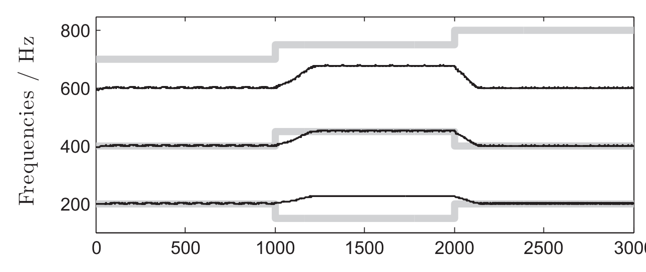


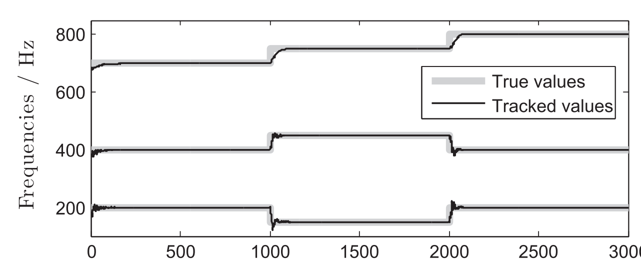
Fig. 10. The original signal $x(n)$ and filtered signals (Case III).



(a) The SMM method



(b) The cascade method



(c) The proposed method

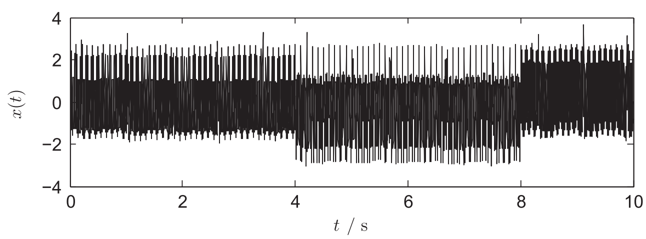
Fig. 11. Tracked frequencies of ANF design methods (Case III).

5. Conclusion

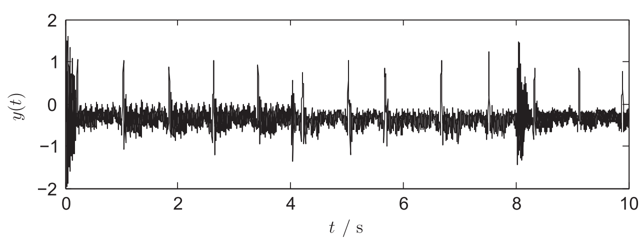
In this paper, a novel design method of an adaptive notch filter with identical bandwidths is proposed to suppress multiple sinusoidal interference components in digital signals. A simplified lin-

Table 3
The interference frequency values in the noisy ECG signal.

t	$0 \sim 4$ s	$4 \sim 8$ s	$8 \sim 10$ s
f_1	50 Hz	40 Hz	50 Hz
f_2	90 Hz	90 Hz	100 Hz
f_3	140 Hz	150 Hz	150 Hz



(a) The input signal $x(t)$.



(b) The filtered signal $y(t)$ via the proposed method

Fig. 12. The input signal and filtered one for ECG interference suppression.

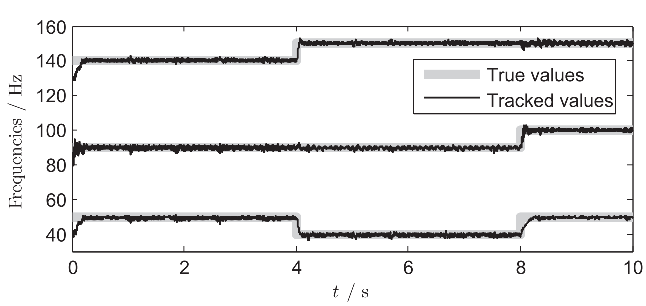


Fig. 13. The tracked frequencies of the proposed method.

ear equation of a non-adaptive notch filter is introduced to decrease the computational complexity. Then, it is applied in the design procedure of adaptive notch filter. Compared with the cascade-based adaptive notch filter, the proposed one can eliminate multiple *non-harmonic* narrow-band components simultaneously. The experimental results validate the effectiveness and practicability of the proposed design method. According to our experiments, the proposed method obtains better design performance than the cascade one in multiple narrow-band interference suppressions, under both harmonic and non-harmonic sinusoidal interference conditions. How to design an adaptive notch filter to track and eliminate much more complex interference component is our major research work in future.

Appendix A. The deduction of (7)

In general, the system function of the notch filter can be constructed as

$$H(z) = \prod_{i=1}^{N} \frac{1 - 2\cos(\omega_i) z^{-1} + z^{-2}}{1 - 2r\cos(\theta_i) z^{-1} + r^2 z^{-2}} \tag{A.1}$$

where ω_i, θ_i, r are notch frequency, polar angle and polar radius, respectively. On the other hand, substituting (3) into (2) yields

$$H(z) = \frac{1 + N(z)/D(z)}{2} = \frac{[D(z) + N(z)]/2}{D(z)} \tag{A.2}$$

Note that (A.1) and (A.2) are two different types of the notch filter representation, and the relationship of them can used to deduct some valuable equations.

By comparing (A.1) with (A.2), $D(z)$ can be represented as

$$D(z) = \prod_{i=1}^{N} (1 - 2r\cos\theta_i z^{-1} + r^2 z^{-2}) \tag{A.3}$$

then applying some manipulations (special skills) to $D(z)$ in (A.3), we have

$$\begin{aligned}(rz^{-1})^{2N} D(r^2 z^{-1}) &= (rz^{-1})^{2N} \prod_{i=1}^{N} (1 - 2r^{-1}\cos\theta_i z + r^{-2} z^2) \\ &= \prod_{i=1}^{N} (1 - 2r\cos\theta_i z^{-1} + r^2 z^{-2}) = D(z)\end{aligned} \tag{A.4}$$

Hence, $D(z) = \sum_{k=0}^{2N} a_k z^{-k}$ can be rewritten as

$$\begin{aligned}\sum_{k=0}^{2N} a_k z^{-k} &= D(z) = (rz^{-1})^{2N} D(r^2 z^{-1}) \\ &= (rz^{-1})^{2N} \sum_{k=0}^{2N} a_k r^{-2k} z^k = \sum_{k=0}^{2N} a_k (r^2)^{N-k} z^{-(2N-k)}\end{aligned} \tag{A.5}$$

Corresponding the polynomial coefficients in (A.5), we have derived (7).

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